Math 33B Worksheet 8 (Higher Dimensional Systems)

Name: ______________________________ Score: __________

Circle the name of your TA: Ziheng Nicholas Victoria

Circle the day of your discussion: Tuesday Thursday

The last two problems on this worksheet are based off of examples from the following link:

`http://www.math.utah.edu/~gustafso/2250systems-de.pdf`

1. Solve $\mathbf{X}' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \mathbf{X}$.

2. Solve $\mathbf{X}' = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{X}$. *Hint:* The general solution is of the form

$$\mathbf{X} = c_1 \mathbf{v}_1 e + c_2 e^{\lambda t}(\mathbf{v}_1 t + \mathbf{v}_2) + c_3 e^{\lambda t} \left(\mathbf{v}_1 \frac{t^2}{2} + \mathbf{v}_2 t + \mathbf{v}_3 \right)$$

where $(A - \lambda I)\mathbf{v}_1 = \mathbf{0}, (A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1, (A - \lambda I)\mathbf{v}_3 = \mathbf{v}_2$.

3. Let A, B, C be brine tanks with volumes 60 L, 30 L, 60 L, respectively. Suppose that fluid drains from tank A to be B at a rate r, drains from tank B to C at rate r, then drains from tank C to A at rate r. The tank volumes remain constant due to constant recycling of the fluid. Take $r = 10$ L/min.

   Let $x_1(t), x_2(t), x_3(t)$ denote the amount of salt at time t in each tank. No salt is lost from the system, due to recycling. The recycled cascade is modeled by the system

$$
\begin{aligned}
x_1' &= -\frac{1}{6}x_1 + \frac{1}{6}x_3 \\
x_2' &= \frac{1}{6}x_1 - \frac{1}{3}x_2 \\
x_3' &= \frac{1}{3}x_2 - \frac{1}{6}x_3.
\end{aligned}
$$

   Find the general solution to this system.

4. Consider a forest having one or two varieties of trees. We select some of the oldest trees, those expected to die off in the next few years, then follow the cycle of living into dead trees. The dead trees eventually decay and fall from seasonal and biological events. Finally, the fallen trees become humus. Let variables x, y, z, t be defined by

$$\begin{aligned} x(t) &= \text{biomass decayed into humus,} \\ y(t) &= \text{biomass of dead trees,} \\ z(t) &= \text{biomass of living trees,} \\ t &= \text{time in decades} \end{aligned}$$

A typical biological model is

$$\begin{aligned} x'(t) &= -x(t) + 3y(t) \\ y'(t) &= -3y(t) + 5z(t) \\ z'(t) &= -5z(t) \end{aligned}$$

Suppose there are no dead trees and no humus at $t = 0$ with initially 80 units of living tree biomass, i.e. $x(0) = y(0) = z(0) = 80$. Find the solution to this system.