Partial Fraction Decomposition Notes & Examples

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We use partial fraction decomposition on rational functions of the form $r(x) = \frac{p(x)}{q(x)}$ where $p(x), q(x)$ are polynomials. Some initial steps:

1. The degree of $p(x)$ **must** be smaller than the degree of $r(x)$. If not, you must perform long division before moving onto the next step.
2. Factor $q(x)$ into linear or irreducible quadratics.

Here are the following cases on how we write the partial fraction decomposition:

I. Linear factors. Linear factors are of the form $mx + b$. We place constants on the top of linear factors.

(i) Nonrepeated linear factors.
Example.

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

(ii) Repeated linear factors.
Example.

$$\frac{1}{(x-1)^2(x+1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$$

II. Irreducible quadratic factors. Irreducible quadratics are of the form $ax^2 + bx + c$ and cannot be factored into linear factors. We place linear terms (e.g. $mx + b$) on the top of quadratic factors.

(i) Nonrepeated irreducible quadratic factors.
Example.

$$\frac{x^3-1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

(ii) Repeated irreducible quadratic factors.
Example.

$$\frac{5x+10}{(x^2+1)^2(x^2+4)^3} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{x^2+4} + \frac{Gx+H}{(x^2+4)^2} + \frac{Ix+J}{(x^2+4)^3}$$

Example.

(a) Find the partial fraction decomposition of $\dfrac{2x^2 - x + 1}{x(x^2+1)}$.

Solution. The degree of the numerator is 2, which is smaller than the degree of the denominator (which is 3), so we don't need to do long division. The denominator is already factored. Note that x is a linear factor, $x^2 + 1$ is an irreducible quadratic factor.

We now write the form of the partial fraction decomposition:

$$\frac{2x^2 - x + 1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

Now we want to find A, B, C. To do this, we multiply both sides by the denominator:

$$\frac{2x^2 - x + 1}{x(x^2+1)} \cdot x(x^2+1) = \frac{A}{x} \cdot x(x^2+1) + \frac{Bx + C}{x^2 + 1} \cdot x(x^2+1)$$

Cancel out the appropriate terms:

$$2x^2 - x + 1 = A(x^2+1) + (Bx + C)x$$

Distribute the right hand side:

$$2x^2 - x + 1 = Ax^2 + A + Bx^2 + Cx$$

Combine like terms:

$$2x^2 - x + 1 = (A + B)x^2 + Cx + A$$

The terms on the left hand side must equal the terms on the right hand side. The x^2 terms on the left hand side must equal the x^2 terms on the right hand side, the x terms on the left hand side must equal the x terms on the right hand side, and so on. This means that the coefficients of these terms must be equal. Here I have highlighted which terms must be matched up:

$$2x^2 - x + 1 = (A + B)x^2 + Cx + A$$

This gives us the following system of equations:

$$2 = A + B$$
$$-1 = C$$
$$1 = A$$

We must solve this system of equations. This system already tells us $A = 1$, $C = -1$. To find B, we can substitute the last equation into the first equation to get $B = 1$.

Now plug A, B, C back into the form of the partial fraction decomposition:

$$\frac{2x^2 - x + 1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

$$\boxed{\frac{2x^2 - x + 1}{x(x^2+1)} = \frac{1}{x} + \frac{x - 1}{x^2 + 1}}$$

□

(b) Use (a) to evaluate $\int \frac{2x^2 - x + 1}{x(x^2+1)} dx$.

Solution. From (a),

$$\int \frac{2x^2 - x + 1}{x(x^2+1)} dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+1} \right) dx$$

Split up the integral:

$$\int \frac{1}{x} dx + \int \frac{x}{x^2+1} dx - \int \frac{1}{x^2+1} dx$$

For the second integral, use the substitution $w = x^2 + 1, dw = 2dx$:

$$\int \frac{1}{x} dx + \frac{1}{2} \int \frac{1}{w} dw - \int \frac{1}{x^2+1} dx$$

Integrate:

$$\ln|x| + \frac{1}{2} \ln|w| - \arctan x + C$$

Plug back in w:

$$\boxed{\ln|x| + \frac{1}{2} \ln|x^2+1| - \arctan x + C}$$

□

Another Example.

(a) Find the partial fraction decomposition of $\frac{2}{(x-1)(x+1)}$.

Solution. The degree of the numerator is 0, which is smaller than the degree of the denominator (which is 2), so we don't need to do long division. The denominator is already factored. Note that $x - 1$, $x + 1$ are both linear factors.

We now write the form of the partial fraction decomposition:

$$\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Now we want to find A, B. Multiply both sides by the denominator:

$$\frac{2}{(x-1)(x+1)} \cdot (x-1)(x+1) = \frac{A}{x-1} \cdot (x-1)(x+1) + \frac{B}{x+1} \cdot (x-1)(x+1)$$

Cancel out the appropriate terms:

$$2 = A(x+1) + B(x-1)$$

Distribute the right hand side:

$$2 = Ax + A + Bx - B$$

Combine like terms:

$$2 = (A + B)x + A - B$$

I'm going to add a placement x on the left hand side:

$$0x + 2 = (A + B)x + A - B$$

The terms on the left hand side must equal the terms on the right hand side. Here I have highlighted which terms must be matched up:

$$0x + 2 = (A + B)x + A - B$$

This gives us the following system of equations:

$$0 = A + B$$
$$2 = A - B$$

We must solve this system of equations. Adding the two equations yields $2A = 2$ or $A = 1$. Substitute A into either equation to get $B = -1$.

Now plug A, B back into the form of the partial fraction decomposition:

$$\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$\boxed{\frac{2}{(x-1)(x+1)} = \frac{1}{x-1} - \frac{1}{x+1}}$$

□

(b) Use (a) to evaluate $\int \frac{2}{(x-1)(x+1)} dx$.

Solution. From (a),

$$\int \frac{2}{(x-1)(x+1)} dx = \int \left(\frac{1}{x-1} - \frac{1}{x+1} \right) dx$$

Split up the integral:

$$\int \frac{1}{x-1} dx - \int \frac{1}{x+1} dx$$

Integrate (use the substitution $w = x - 1$ for the first integral, $z = x + 1$ for the second integral):

$$\boxed{\ln |x - 1| - \ln |x + 1| + C}$$

□