Math 31AL Notes

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Limits and Continuity

- If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$, then the limit $\lim_{x\to a} f(x)$ exists.
- If $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then the limit $\lim_{x\to a} f(x)$ does not exist (DNE).
- If the limit $\lim_{x\to a} f(x) = f(a)$, then we say that f is continuous at a.

Some Special Limits

$$\lim_{x\to\pm\infty} \frac{1}{x} = 0$$

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

Strategies for Evaluating Limits

1. Plug in the limit.
2. Is the limit of the form $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$ (i.e. an indeterminate form)?
   If no: you are done!
   If yes: proceed to the next steps.
3. If the limit involves a lot of fractions, write as one fraction.
4. If the limit is going to $\pm\infty$ and involves powers, polynomials, or roots, e.g.

$$\lim_{x\to\pm\infty} \frac{x^2+1}{7-x^2} \quad \text{or} \quad \lim_{x\to\pm\infty} \frac{\sqrt{x^2+3x}}{x^4-x^2+1}$$

then multiply the numerator and denominator by 1 / (highest power in numerator) or 1 / (highest power in denominator). With square roots, you may also need to use

$$\frac{1}{x} = \begin{cases} \frac{1}{\sqrt{x^2}}, & x \geq 0 \\ -\frac{1}{\sqrt{x^2}}, & x < 0. \end{cases}$$

5. If the limit involves polynomials over polynomials, e.g.

$$\lim_{x \to 1} \frac{x-1}{x^2-1},$$

   try factoring. You may need to use long division.

6. If the limit involves sin or cos, then

   (a) Try to use the special limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \text{or} \quad \lim_{x \to 0} \frac{1-\cos x}{x} = 0$$

   (b) Use Squeeze Theorem.

7. If the limit involves other trigonometric functions like tan, sec, csc, cot, write the limit in terms of sin and cos and use the methods in the previous step.

8. If the limit involves square roots with addition and subtraction, e.g.

$$\lim_{x \to 16} \frac{4-\sqrt{x}}{16-x}$$

   use the conjugate.

Squeeze Theorem

If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$ and

$$f(x) \leq g(x) \leq h(x),$$

then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \leq \lim_{x \to a} h(x)$$

which implies

$$\lim_{x \to a} g(x) = \lim_{x \to a} f(x) = \lim_{x \to a} h(x).$$

Some useful inequalities:

$$-1 \leq \sin x \leq 1$$

$$-1 \leq \cos x \leq 1$$

$$-1 \leq \frac{x}{|x|} \leq 1$$

Intermediate Value Theorem

If $f(x)$ is continuous (i.e. no jumps, asymptotes, breaks) and $f(a) > 0$, $f(b) < 0$, then there is a number c between a and b such that $f(c) = 0$.

Derivatives

The formal definition of the derivative of $f(x)$ at a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

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Some special derivatives:

$$\frac{d}{dx} c = 0, \quad \text{where } c \text{ is a constant}$$

$$\frac{d}{dx} x^n = nx^{n-1}, \quad n \neq 0$$

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x), \quad \text{“Product Rule”}$$

$$\left(\frac{T(x)}{B(x)}\right)' = \frac{B(x)T'(x) - T(x)B'(x)}{(B(x))^2}, \quad \text{“Quotient Rule”}$$

$$(f(g(x)))' = f'(g(x))g'(x), \quad \text{“Chain Rule”}$$

Tangent Lines

To find a tangent line of a function $f(x)$ through the point $(a, f(a))$:

1. Find $f'(x)$.
2. Plug in a into the derivative, i.e. $f'(a)$, this is your slope.
3. Use the formula $y - f(a) = f'(a)(x - a)$.

Implicit Differentiation

We use this whenever we have to take the derivative of something involving both x and y. Use the following steps:

1. Take the derivative of both sides. Make sure to multiply by y' whenever you take the derivative of something involving y.
2. Solve for y'.

Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable (a, b), then there exists at least one value c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Extreme Values

To find the **local extrema** of a function $f(x)$:

1. Find $f'(x)$.
2. Find the critical points of f by setting $f'(x) = 0$ (if f' has a denominator, you must also set the denominator equal to 0).
3. Use either the First Derivative Test or Second Derivative Test to classify the critical points as a local maxmima or minima.
   - **First Derivative Test:**
     (i) Draw a number line with the critical points.
     (ii) Test points on the number line. If $f'(x) > 0$, the function is increasing. If $f'(x) < 0$, the function is decreasing.
     (iii) Increasing to decreasing indicates a local maximum, decreasing to increasing indicates a local minimum.
   - **Second Derivative Test:** For each of the critical points, evaluate $f''(x)$. If $f''(x) > 0$, the function is concave up and there is a local minimum. If $f''(x) < 0$, the function is concave down and there is a local maximum.

To find the **absolute extrema** of a function $f(x)$ on an interval $[a, b]$:

1. Find the local extrema.
2. Evaluate f at the local extrema and $f(a), f(b)$. Whichever is the highest value is the absolute maximum, whichever is the smallest value is the absolute minimum.

Graphing Functions

Use the following steps to graph a function (some steps may not always be necessary):

1. Find the x-intercepts (set $f(x) = 0$) and y-intercepts (evaluate $f(0)$).
2. Find the domain of the function. If the function has a denominator involving x, this usually indicates vertical asymptotes.
3. Use the first derivative to find the critical points, where the function is increasing and decreasing, and local extrema.
4. Use the second derivative determine to find where the function is concave up or concave down.
5. Find the long term behavior by evaluating $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
6. Plot all of this information on one graph. If necessary, plug in some points.

Calculating Areas

To approximate the area under the graph of f over the interval $[a, b]$, calculate

$$\Delta x = \frac{b-a}{N} \quad \text{and} \quad x_j = a + j\Delta x$$

and

$$L_N = \Delta x \sum_{j=0}^{N-1} f(x_j)$$

$$R_N = \Delta x \sum_{j=1}^{N} f(x_j)$$

$$M_N = \Delta x \sum_{j=0}^{N-1} f\left(\frac{x_j + x_{j+1}}{2}\right).$$

The area under the graph of $f(x)$ over the interval $[a, b]$ is given by

$$\lim_{N\to\infty} L_N \quad \text{or} \quad \lim_{N\to\infty} R_N \quad \text{or} \quad \lim_{N\to\infty} M_N.$$