Tuesday, November 5

1. $f(x) = x^2 - 2x^2 + 2x - 12$
   - Increase "increasing" by finding the derivative
   - $f'(x) = 3x^2 - 4x + 2$
   - Try to find the critical points, i.e., $f'(x) = 0$: $3x^2 - 4x + 2 = 0$
     - Quadratic formula: $x = \frac{4 \pm \sqrt{(-4)^2 - 4(3)(2)}}{2(3)}$
     - $x = \frac{4 \pm \sqrt{16 - 24}}{6}$
     - No real solution, no critical points.
   - $f'(x)$ doesn't equal 0 -> either $f'(x) > 0$ (increasing) or $f'(x) < 0$ (decreasing).
   - Plug in random point: $f'(0) = 2 > 0$, so $f'(x) > 0$ all the time. 

2. $f(x) = x^3 + 9x - 4$
   - Intermediate Value Theorem says if $f$ continuous, then there is a $c$ between $a$ and $b$ s.t. $f(c) = 0$.
   - $f(x)$ is a polynomial, so it is continuous.
   - Pick a number a s.t. $f(a) > 0$, e.g., $f(1) = 1^3 + 9(1) - 4 = 1 + 9 = 10 > 0$
   - Pick a number b s.t. $f(b) < 0$, e.g., $f(0) = 0^3 + 9(0) - 4 = -4 < 0$
   - So by IVT, there is a number $c$ between 0 and 1 s.t. $f(c) = 0$ (i.e., a root). 

3. Answers will vary.
   - ![Graph 1](attachment:graph1.png)
   - ![Graph 2](attachment:graph2.png)

4. $y = 3x^4 + 8x^2 - 6x^2 - 24x$
   - $y' = 12x^3 + 24x - 12x - 24$
   - Plug in 1, get $y' = 12(1)^3 + 24(1) - 12(1) - 24 = 12 + 24 - 12 - 24 = 0$
   - So $x = 1$ is a factor of $12x^3 + 24x - 12x - 24$
   - Use long division or synthetic division to factor:
     - $x - 1$ $| 12x^3 + 24x - 12x - 24$
     - $-(12x^3 - 12x)$
     - $\underline{36x^2 - 12x}$
     - $-(36x^2 - 36x)$
     - $24x - 24$
     - $-(24x - 24)$
     - $0$
   - So $y' = (x-1)(12x^2 + 36x + 24)$
   - $y' = 12(x-1)(x^2 + 3x + 2)$
   - $y' = 12(x-1)(x+2)(x+1)$
   - Critical points occur when $y' = 12(x-1)(x+2)(x+1) = 0$
     - $x = 0, x = -2, x = -1$
Thursday, Nov 7

4. continued

\[
y = 12(x-1)(x+3)(x+1)
\]

Test points,
\[
y'(-3) = 12(-3-1)(-3+3)(-3+1) = 12(-4)(-1) = 48 < 0 \quad \text{dec}
\]
\[
y'(-1) = 12(-1-1)(-1+3)(-1+1) = 12(-2)(0) = 0 \quad \text{inc}
\]
\[
y'(0) = 12(0-1)(0+3)(0+1) = 12(-1)(3) = 0 < 0 \quad \text{dec}
\]
\[
y'(3) = 12(3-1)(3+3)(3+1) = 12(2)(4) > 0 \quad \text{inc}
\]

\[
y \quad \text{is increasing on (-2,-1) U (1,\infty)}
\]
\[
\text{decreasing on } (-\infty,-2) \text{ U (-1,1)}
\]

Local max occurs when inc -> dec so local max at \( X = -1 \)

Local min occurs when dec -> inc so local min at \( X = -2, 1 \)

Plug into original function:
\[
y = 2x^3 + 3x^2 - 6x - 24
\]
\[
y'(-3) = 3(9-6) + 2(-1) - 3 - 24 = 3 - 8 - 24 = -25
\]
\[
y'(-2) = 3(4-6) + 3(4-2) - 2(4) - 2(2) = 12 - 6 - 4 - 4 = 0
\]
\[
y'(0) = 3(0)^3 - 6(0)^2 - 24(0) = 3 + 0 - 24 = -21
\]
\[
y'(1) = 3(1)^3 + 3(1)^2 - 6(1) - 24(1) = 3 + 3 - 6 - 24 = -18
\]
\[
\begin{align*}
\text{(-1/3) local max} \\
\text{(-2,1) local min} \\
\text{(1,19) local min}
\end{align*}
\]

5. \( f(x) = x^3 + ax + b \)

\[
f'(x) = 3x^2 + a \quad \text{Want } f''(x) > 0 \quad \text{always,}
\]

so \( f'(x) = 3x^2 + a > 0 + a = a \). So we need \( a > 0 \) for \( f'' \) to be increasing. No restrictions on \( b \).

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Given \( f''(x) = 30(x-\frac{3}{5}) \)

Find critical points: \( 30(x-\frac{3}{5}) = 0 \)

\[
\begin{align*}
\text{X-3/5} & = 0 \\
X & = \frac{3}{5}
\end{align*}
\]

Test points: \( f'(\frac{3}{5}) = 20(\frac{3}{5}-\frac{3}{5}) = 20(\frac{6}{5}) = \frac{120}{5} = + + = + \\
(\frac{3}{5} - \frac{2}{5}) = 20(\frac{1}{5}) = \frac{20}{5} = + + = + \\
(\frac{4}{5} - \frac{3}{5}) = 20(\frac{1}{5}) = \frac{20}{5} = + + = + \\
(\frac{5}{5} - \frac{4}{5}) = 20(\frac{1}{5}) = \frac{20}{5} = + + = +
\]

Inc -> dec means local max, local max at \( x = \frac{3}{5} \)

Dec -> inc means local min, local min at \( x = \frac{3}{5} \)

Given \( f''(x) = 20(\frac{x}{5}) \)

Set numerator = 0: \( 20(\frac{x}{5}) = 0 \)

\( x = \frac{3}{5} \)
Test point: \( f''(0) = \frac{20(0-3)}{(0-1)^2} = \frac{20(-\frac{3}{2})}{(-1)^2} = -\frac{30}{4} = -7.5 \) 

\( f''(\frac{1}{2}) = \frac{20(\frac{1}{2}-\frac{3}{2})}{(\frac{1}{2}-1)^2} = \frac{20(\frac{1}{2})}{(-\frac{1}{2})^2} = +10 = + \) 

\( f''(2) = \frac{20(2-\frac{3}{2})}{(2-1)^2} = \frac{20(\frac{1}{2})}{1} = +10 = + \) 

\( \wedge \) to \( U \) means inflection point at \( x = \frac{3}{2} \) 

\( f''(\frac{3}{2}) = 18(\frac{3}{2} - 3)(\frac{3}{2} - 1)^{\frac{3}{2}} \approx -28.5 \) inflection pt at \( (\frac{3}{2}, -28.5) \)

Asymptotic behavior: 
\[ \lim_{x \to \infty} f(x) = \lim_{x \to -\infty} 18 (x-3)(x-1)^{\frac{3}{2}} = \infty \] 
\[ \lim_{x \to 0} f(x) = \lim_{x \to \infty} 18 (x-3)(x-1)^{\frac{3}{2}} = -\infty \] 

\( \lim_{x \to -\infty} f(x) = \lim_{x \to \infty} 18 (x-3)(x-1)^{\frac{3}{2}} = \text{number} \)

1. continued

2. on next worksheet