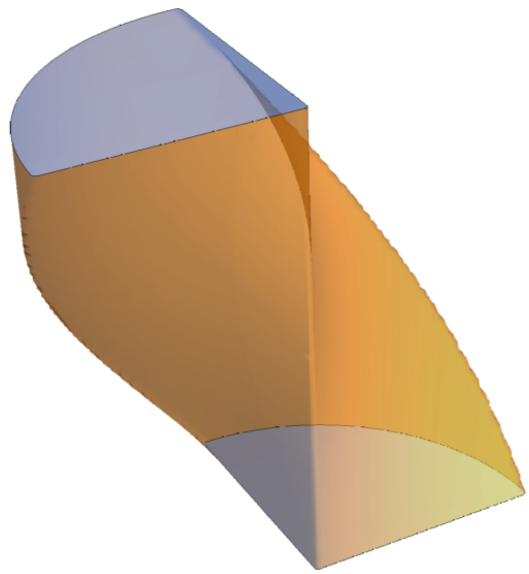
Problem 1 Let $\mathcal{D}$ be the region in $\mathbb{R}^3$ bounded by the surfaces (described in cylindrical coordinates) $\theta = z$, $\theta = z + \pi/2$, $z = 0$, $z = \pi/2$, and $r = 1$.

(a) Sketch the region $\mathcal{D}$.

Solution. $\mathcal{D}$ is a "twisted quarter cylinder": each horizontal cross section is a quarter circle, and these quarter circles rotate from the first quadrant (at the bottom of the cylinder) to the second quadrant (at the top of the cylinder). Your sketch might look something like this:



Animations of $\mathcal{D}$ being built from rotating quarter circles can be found here: `https://imgur.com/a/N5bqz5L`.

(b) Find the volume of $\mathcal{D}$.

Solution. Note that the region $\mathcal{D}$ is very easy to describe in cylindrical coordinates: the bounds on r are simply $0 \leq r \leq 1$, and for a fixed value of r, the cross section $\mathcal{D}$ of in z-θ space is a θ-simple parallelogram. Thus, one way to compute the volume is with the following triple integral:

$$\int_0^{\pi/2} \int_z^{z+\pi/2} \int_0^1 r \, \mathrm{d}r \mathrm{d}\theta \mathrm{d}z = \frac{1}{2} \int_0^{\pi/2} \int_z^{z+\pi/2} \mathrm{d}\theta \mathrm{d}z = \frac{\pi}{4} \int_0^{\pi/2} \mathrm{d}z = \boxed{\frac{\pi^2}{8}}$$

Another way to arrive at this answer is to note that the volume of $\mathcal{D}$ must be exactly one fourth the volume of the cylinder of radius 1 and height $\pi/2$, since four copies of $\mathcal{D}$ fit together to make such a cylinder. A cylinder of radius 1 and height $\pi/2$ has volume $\pi^2/2$, so $\mathcal{D}$ has volume $\pi^2/8$.

Yet another way to arrive at this answer is to note that each horizontal cross section of $\mathcal{D}$ is a quarter circle of area $\pi/4$, so the total volume of $\mathcal{D}$ must be $(\pi/4)(\pi/2) = \pi^2/8$.

(c) Compute $\iiint_{\mathcal{D}} xyz \, \mathrm{d}V$.
Solution. In cylindrical coordinates, the function becomes $r^2 \cos(\theta)\sin(\theta)z = r^2 \sin(2\theta)z/2$. Thus, we need to evaluate

$$\frac{1}{2}\int_0^{\pi/2}\int_z^{z+\pi/2}\int_0^1 r^3 \sin(2\theta) z \, \mathrm{d}r\mathrm{d}\theta\mathrm{d}z = \frac{1}{8}\int_0^{\pi/2} z \left(\int_z^{z+\pi/2} \sin(2\theta)\, \mathrm{d}\theta\right) \mathrm{d}z.$$

First,

$$\int_z^{z+\pi/2} \sin(2\theta)\, \mathrm{d}\theta = [-\cos(2\theta)/2]_z^{z+\pi/2}$$
$$= \frac{1}{2}(\cos(2z) - \cos(2z+\pi)) = \cos(2z),$$

so the integral becomes

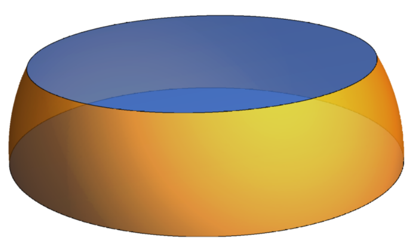
$$\frac{1}{8}\int_0^{\pi/2} z\cos(2z)\, \mathrm{d}z.$$

Letting $u = z$ and $v = \sin(2z)/2$, this is

$$\frac{1}{8}\int_{z=0}^{\pi/2} u\, \mathrm{d}v = \frac{1}{8}\left([uv]_{z=0}^{\pi/2} - \int_{z=0}^{\pi/2} v\, \mathrm{d}u\right) = \frac{1}{8}\left(\frac{-1}{2}\int_0^{\pi/2}\sin(2z)\, \mathrm{d}z\right)$$
$$= \frac{-1}{16}\left[\frac{-\cos(2z)}{2}\right]_0^{\pi/2} = \boxed{\frac{-1}{16}}$$

Problem 2 For each $h \in [0,1]$, let $\mathcal{R}_h$ be the region in $\mathbb{R}^3$ bounded by the surfaces $z = 0$, $z = h$, and $x^2 + y^2 + z^2 = 1$.

(a) Sketch the regions $\mathcal{R}_1$ and $\mathcal{R}_{\frac{1}{2}}$.
Solution. $\mathcal{R}_1$ is simply a half ball, and $\mathcal{R}_{1/2}$ is part of a half ball, cut off at $z = 1/2$ as shown below:



(b) Let $f(h) = \iiint_{\mathcal{R}_h} \mathrm{d}V$. Explain why $f'(h) = \pi(1-h^2)$.
Solution. Each horizontal cross-section of $\mathcal{R}_h$ is a circle, so intuitively $f'(h)$ should be the area of the circle which is the top face of $\mathcal{R}_h$. The top face of $\mathcal{R}_h$ is defined by $z = h$ and $x^2 + y^2 \leq \sqrt{1-z^2}$, so the area of this circle is $\pi(1-h^2)$.

Alternatively, it's easy to check that

$$f(h) = \int_0^h \int_0^{2\pi} \int_0^{\sqrt{1-z^2}} r \, \mathrm{d}r\mathrm{d}\theta\mathrm{d}z,$$

so by the fundamental theorem of calculus,

$$f'(h) = \int_0^{2\pi} \int_0^{\sqrt{1-h^2}} r \, \mathrm{d}r\mathrm{d}\theta\mathrm{d}z = \pi(1-h^2).$$

(c) Find $f(h)$ by evaluating the integral $\iiint_{\mathcal{R}_h} \mathrm{d}V$ using cylindrical coordinates.

Solution. Using cylindrical coordinates, we have

$$\iiint_{\mathcal{R}_h} \mathrm{d}V = \int_0^h \int_0^{2\pi} \int_0^{\sqrt{1-z^2}} r \, \mathrm{d}r\mathrm{d}\theta\mathrm{d}z = \int_0^h \pi(1-z^2) \, \mathrm{d}z = \pi(h-h^3/3).$$

(d) Find $f(h)$ by evaluating the integral $\iiint_{\mathcal{R}_h} \mathrm{d}V$ using spherical coordinates.

Solution. The tricky thing about this problem is that the region is not ρ-simple. To get around this, instead of finding the volume of $\mathcal{R}_h$ directly, we'll find the volume of $\mathcal{R}_1$ (the half ball) and subtract the extra "cap" at the top. Let $\mathcal{R}'_h$ be the extra "cap": it is bounded by $x^2+y^2+z^2=1$ and $z=h$. The good is news is that $\mathcal{R}'_h$ is ρ-simple, so its volume is easier to find with spherical coordinates: the boundary $z=h$ can be expressed in spherical coordinates as $\rho\cos\phi = h$, or in other words $\rho = h\sec\phi$. Thus, our bounds for ρ are $h\sec\phi \le \rho \le 1$. $\mathcal{R}'_h$ is symmetric around the z-axis, so our bounds for θ are $0 \le \theta \le 2\pi$. Because $\mathcal{R}'_h$ includes points on the z-axis, the lower bound on ϕ is 0. The largest value of ϕ attained on $\mathcal{R}'_h$ occurs on the boundary of the bottom face, where $z=h$ and $\rho=1$. This gives an upper bound on ϕ of $\arccos(h)$. Thus,

$$\begin{aligned}
\iiint_{\mathcal{R}'_h} \mathrm{d}V &= \int_0^{\arccos h} \int_0^{2\pi} \int_{h\sec\phi}^1 \rho^2 \sin\phi \, \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{1}{3}\int_0^{\arccos h} \int_0^{2\pi} (1-h^3\sec^3\phi)\sin\phi \, \mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{1}{3}\left(\int_0^{\arccos h} \int_0^{2\pi} \sin\phi \, \mathrm{d}\theta\mathrm{d}\phi - h^3 \int_0^{\arccos h} \int_0^{2\pi} \sec^3\phi\sin\phi \, \mathrm{d}\theta\mathrm{d}\phi\right) \\
&= \frac{2\pi}{3}\left(\int_0^{\arccos h} \sin\phi \, \mathrm{d}\phi - h^3 \int_0^{\arccos h} \sec^2\phi\tan\phi \, \mathrm{d}\phi\right) \\
&= \frac{2\pi}{3}\left(1-h-h^3 \int_0^{\arccos h} \sec^2\phi\tan\phi \, \mathrm{d}\phi\right).
\end{aligned}$$

We will now focus on the integral

$$\int_0^{\arccos h} \sec^2\phi\tan\phi\mathrm{d}\phi.$$

Letting $u = \tan\phi$, we have $\mathrm{d}u = \sec^2\phi\mathrm{d}\phi$, so the integrand above is $u\mathrm{d}u$. When $\phi = 0$, $u = \tan(0) = 0$, and when $\phi = \arccos h$, $u = \tan(\arccos h) = \frac{\sqrt{1-h^2}}{h}$ (see Figure 1 below). Thus, this integral is just

$$\int_0^{\sqrt{1-h^2}/h} u \ \mathrm{d}u = \frac{1-h^2}{2h^2}.$$

Therefore, the volume of $\mathcal{R}'_h$ is

$$\frac{2\pi}{3}\left(1 - h - h^3\frac{1-h^2}{2h^2}\right) = \frac{\pi}{3}(2 - h + h^3).$$

$\mathcal{R}_1$ is half of a unit ball, so its volume is $(4\pi/3)/2 = 2\pi/3$. Thus, the volume of $\mathcal{R}_h$ is

$$\frac{2\pi}{3} - \frac{\pi}{3}(2 - h + h^3) = \boxed{\frac{\pi}{3}(h - h^3)},$$

just like before.

(e) Evaluate

$$\iiint_{\mathcal{R}_{1/2}} \frac{z}{\sqrt{x^2+y^2+z^2}} \ \mathrm{d}V.$$

Solution. The function is easy to write in spherical coordinates: it becomes simply $\rho\cos\phi/\rho = \cos\phi$. As a result, it makes sense to do this integral in spherical coordinates. Just like in Problem (2.d), we'll compute this by integrating over $\mathcal{R}_1$ and subtracting the integral over the "extra cap". First, the integral over the entire half ball $\mathcal{R}_1$ is

$$\int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \rho^2\sin\phi\cos\phi \ \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi = \frac{\pi}{3}.$$

Next, the integral over the extra cap is

$$\begin{aligned}
&\int_0^{\arccos\frac{1}{2}}\int_0^{2\pi}\int_{\sec\phi/2}^1 \rho^2\sin\phi\cos\phi \ \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi \\
&\qquad = 2\pi\int_0^{\pi/3}\int_{\sec\phi/2}^1 \rho^2\sin\phi\cos\phi \ \mathrm{d}\rho\mathrm{d}\phi \\
&\qquad = \frac{2\pi}{3}\int_0^{\pi/3}\left(1 - \frac{\sec^3\phi}{8}\right)\sin\phi\cos\phi \ \mathrm{d}\phi \\
&\qquad = \frac{2\pi}{3}\left(\int_0^{\pi/3}\sin\phi\cos\phi \ \mathrm{d}\phi - \frac{1}{8}\int_0^{\pi/3}\frac{\sin\phi}{\cos^2\phi} \ \mathrm{d}\phi\right).
\end{aligned}$$

In the first integral, setting $u = \sin\phi$ gives $\mathrm{d}u = \cos\phi\mathrm{d}\phi$, thus reducing the integral to $\int_0^{\sqrt{3}/2} u \ \mathrm{d}u$. In the second integral, setting $u = -\cos\phi$

gives $\mathrm{d}u = \sin\phi \mathrm{d}\phi$, thereby reducing the integral to $\int_{-1}^{-1/2} \frac{\mathrm{d}u}{u^2} = \int_{1/2}^{1} \frac{\mathrm{d}u}{u^2}$. In total, we have

$$\frac{2\pi}{3}\left(\int_0^{\sqrt{3}/2} u \, \mathrm{d}u - \frac{1}{8}\int_{1/2}^{1} \frac{\mathrm{d}u}{u^2}\right) = \frac{2\pi}{3}\left(\frac{3}{8} - \frac{1}{8}\right) = \frac{\pi}{6}.$$

Thus, our final answer is

$$\frac{\pi}{3} - \frac{\pi}{6} = \boxed{\frac{\pi}{6}}$$

Problem 3 Let $\mathcal{E}$ be the region defined by $x \geq 0$, $y \leq x$, $z \geq 0$, $x^2 + y^2 \leq 4$, and $xz \leq y$. Evaluate

$$\iiint_{\mathcal{E}} (z - x^2 - y^2) \, \mathrm{d}V.$$

Solution. This region is fairly easy to describe in cylindrical coordinates: the bounds are $0 \leq \theta \leq \pi/4$, $0 \leq r \leq 2$, and $0 \leq z \leq \tan\theta$. The function is also easy to write in cylindrical coordinates: it's just $z - r^2$. Thus, it makes sense to do this integral in cylindrical coordinates. We get

$$\int_0^2 \int_0^{\pi/4} \int_0^{\tan\theta} (z - r^2) r \, \mathrm{d}z\mathrm{d}\theta\mathrm{d}r = \int_0^2 \int_0^{\pi/4} \left(\frac{r\tan^2\theta}{2} - r^3\tan\theta\right) \mathrm{d}\theta\mathrm{d}r.$$

First, recalling that $\tan^2\theta = \sec^2\theta - 1$, we get

$$\int_0^{\pi/4} \tan^2\theta \, \mathrm{d}\theta = \int_0^{\pi/4} \sec^2\theta \, \mathrm{d}\theta - \frac{\pi}{4} = \tan(\pi/4) - \tan(0) - \frac{\pi}{4} = 1 - \frac{\pi}{4}.$$

This tells us that

$$\int_0^2 \frac{r\tan^2\theta}{2} \, \mathrm{d}\theta\mathrm{d}r = \frac{4-\pi}{8}\int_0^2 r \, \mathrm{d}r = 1 - \frac{\pi}{4}.$$

Next, letting $u = \cos\theta$, we get $\mathrm{d}u = -\sin\theta\mathrm{d}\theta$, so $\tan\theta\mathrm{d}\theta = \frac{-\mathrm{d}u}{u}$. Now

$$\int_0^2 \int_0^{\pi/4} r^3\tan\theta \, \mathrm{d}\theta\mathrm{d}r = 4\int_1^{1/\sqrt{2}} \frac{-\mathrm{d}u}{u} = 4\int_{1/\sqrt{2}}^{1} \frac{\mathrm{d}u}{u}$$
$$= 4(\ln 1 - \ln 2^{-1/2}) = 2\ln 2.$$

Thus, our final answer is

$$\boxed{1 - \frac{\pi}{4} - 2\ln 2}$$

Problem 4 Let $\mathcal{E}$ be the region in the first octant bounded by $x^2 + y^2 + z^2 = 4$.

(a) Evaluate

$$\iiint_{\mathcal{E}} (x - y + 3z) \, \mathrm{d}V.$$

Solution. Since $\mathcal{E}$ is symmetric about the plane $x = y$, the integrals $\iiint_{\mathcal{E}} x \text{ d}V$ and $\iiint_{\mathcal{E}} y \text{ d}V$ must be equal. Thus, the integral in question is equal to $3\iiint_{\mathcal{E}} z \text{ d}V$. We will perform this integral in spherical coordinates, because both the domain of integration and the integrand are reasonably easy to describe in spherical coordinates. We have

$$\iiint_{\mathcal{E}} z \text{ dV} = \int_0^{\pi/2}\int_0^1\int_0^{\pi/2} (\rho\cos\phi)(\rho^2\sin\phi) \text{ d}\phi\text{d}\rho\text{d}\theta$$
$$= \int_0^{\pi/2}\int_0^1 \rho^3\left(\int_0^{\pi/2}\cos\phi\sin\phi \text{ d}\phi\right) \text{d}\rho\text{d}\theta$$
$$= \frac{1}{2}\int_0^{\pi/2}\int_0^1 \rho^3 \text{ d}\rho\text{d}\theta = \frac{1}{8}\int_0^{\pi/2} \text{d}\theta = \frac{\pi}{16}.$$

Thus, our final answer is

$$\boxed{\frac{3\pi}{16}}$$

(b) Evaluate

$$\iiint_{\mathcal{E}} (x^2 + y^2 - z^2) \text{ d}V.$$

Solution. Note that $\iiint_{\mathcal{E}}(x^2 + y^2 + z^2) \text{ d}V$ is very easy to compute in spherical coordinates: it's simply

$$\int_0^{\pi/2}\int_0^1\int_0^{\pi/2} \rho^4\sin\phi \text{ d}\phi\text{d}\rho\text{d}\theta = \frac{\pi}{10}.$$

The integral we want to find can be computed by subtracting $\iiint_{\mathcal{E}} 2z^2 \text{ d}V$ from this quantity, so we just need to find $\iiint_{\mathcal{E}} z^2 \text{ d}V$. We compute this in spherical coordinates, as follows:

$$\iiint_{\mathcal{E}} z^2 \text{ dV} = \int_0^{\pi/2}\int_0^1\int_0^{\pi/2} \rho^4\cos^2\phi\sin\phi \text{ d}\phi\text{d}\rho\text{d}\theta$$
$$= \frac{\pi}{10}\int_0^{\pi/2}\cos^2\phi\sin\phi \text{ d}\phi = \frac{-\pi}{10}\int_1^0 u^2 \text{ d}u = \frac{\pi}{30},$$

where is the u-substitution performed was $u = \cos\phi, \text{d}u = -\sin\phi\text{d}\phi$. Thus, our final answer is

$$\frac{\pi}{10} - 2\frac{\pi}{30} = \boxed{\frac{\pi}{30}}$$

(c) Griff makes a piece of Jell-O® whose shape is $\mathcal{E}$. Because Griff isn't very good at cooking, the density of her Jello-O® is very uneven: it is given by the function $\delta(x, y, z) = \frac{x^2+y^2}{x^2+y^2+z^2}$. Find the mass of Griff's Jell-O®.

Solution. The mass of the Jell-O® is simply the integral of the density. The density function is most easily written in spherical coordinates: it is simply $\rho^2 \sin^2 \phi / \rho^2 = \sin^2 \phi$. Thus, we evaluate

$$\int_0^{\pi/2} \int_0^1 \int_0^{\pi/2} \rho^2 \sin^3 \phi \, \mathrm{d}\phi \mathrm{d}\rho \mathrm{d}\theta = \frac{\pi}{6} \int_0^{\pi/2} \sin^3 \phi \, \mathrm{d}\phi.$$

We now use integration by parts. Letting $u = \sin^2 \phi$ and $v = -\cos \phi$, we get $\mathrm{d}u = \sin \phi \cos \phi \mathrm{d}\phi$ and $\mathrm{d}v = \sin \phi \mathrm{d}\phi$. Our integral then becomes

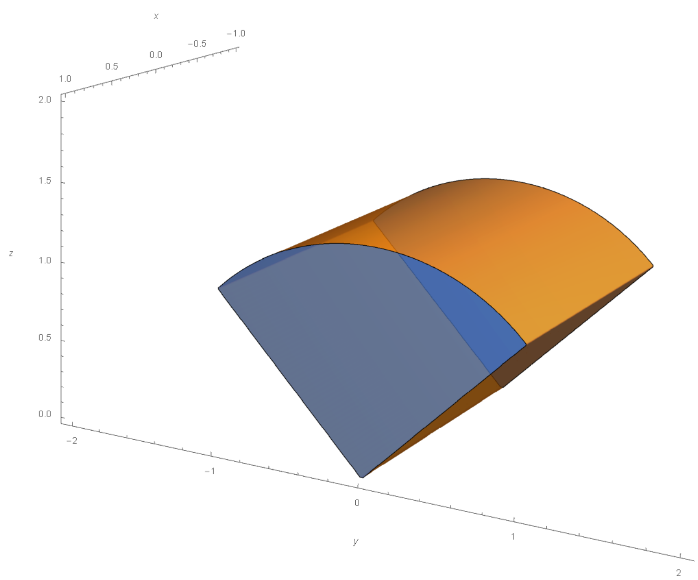
$$\frac{\pi}{6} \int_{\phi=0}^{\pi/2} u \, \mathrm{d}v = \frac{\pi}{6} \left([-\sin^2 \phi \cos \phi]_{\phi=0}^{\pi/2} + \int_{\phi=0}^{\pi/2} \cos^2 \phi \sin \phi \, \mathrm{d}\phi \right).$$

We already found in Problem (4.b) that $\int_{\phi=0}^{\pi/2} \cos^2 \phi \sin \phi \, \mathrm{d}\phi = 1/3$, so our final answer is

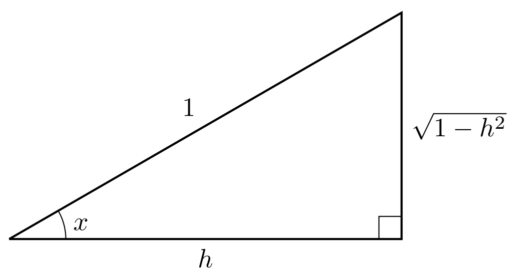
$$\boxed{\frac{\pi}{18}}$$

Problem 5 Let $\mathcal{S}$ be the region bounded by $x = -1$, $x = 1$, $y + z = 0$, $y - z = 0$, and $y^2 + z^2 = 2$. Find the volume of $\mathcal{S}$ (Hint: sketch what S looks like and think about which coordinate system to use).

Solution. $\mathcal{S}$ is a quarter of a cylinder of height 2 and radius $\sqrt{2}$, as shown:



Therefore, its volume is π. If you'd like to compute the volume explicitly via an integral, it's easiest to use modified cylindrical coordinates, where r and θ describe a position in the yz-plane, and the x-coordinate plays the role of the traditional z-coordinate in standard cylindrical coordinates. Then the bounds become $1 \leq x \leq -1$, $0 \leq r \leq \sqrt{2}$, and $-\pi/4 \leq \theta \leq \pi/4$.



Consider the right triangle above. Since $\cos x = h$, $x = \arccos(h)$. Thus,

$$\boxed{\tan(\arccos(h)) = \tan(x) = \frac{\sqrt{1-h^2}}{h}}$$

Figure 1: Computing $\tan(\arccos(h))$.