Problem 1 Sketch the following planar vector fields $\vec{F}$.

(a) $\vec{F} = \langle x^2, y\rangle$

(b) $\vec{F} = \langle y, x+2\rangle$

Problem 2 Calculate $\nabla \times \vec{F}$ and $\nabla \cdot \vec{F}$ of the following:

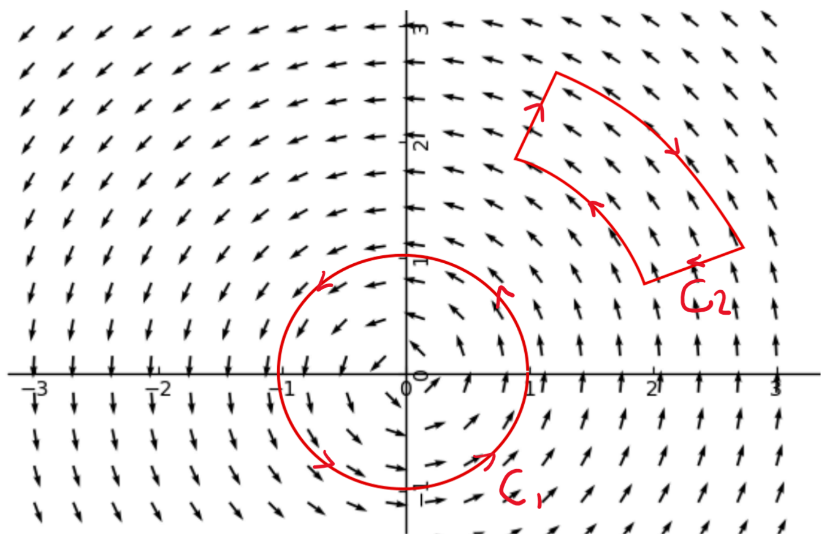
(a) $\vec{F} = \langle y, z, x\rangle$

(b) $\vec{F} = \left\langle \dfrac{-y}{\sqrt{x^2+y^2}}, \dfrac{x}{\sqrt{x^2+y^2}}, 0 \right\rangle$

Problem 3 Show that $f(x,y,z) = r = \sqrt{x^2+y^2+z^2}$ is a potential function for the unit radial vector field $\vec{e}_r = \left\langle \dfrac{x}{r}, \dfrac{y}{r}, \dfrac{z}{r} \right\rangle$

Problem 4 Let $\vec{F} = \left\langle \dfrac{-y}{\sqrt{x^2+y^2}}, \dfrac{x}{\sqrt{x^2+y^2}} \right\rangle$. This is sometimes known as the unit vortex field. We will investigate some properties of it.

(a) In the figure below, we have the vector field and two curves C_1 and C_2. Use a geometric argument to determine whether $\int_C \vec{F} \cdot d\vec{r}$ is positive, negative or zero for the two curves.



(b) The circle C_1 is the unit circle around the origin. Calculate $\int_{C_1} \vec{F} \cdot d\vec{r}$ explicitly by giving C_1 a parameterisation.

Problem 5 For this question, let $\vec{F} = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$. This is almost the same as the previous question except we've multiplied by a factor of $\dfrac{1}{r}$. This causes the length of each vector to get bigger as we approach the origin and get smaller the further from the origin we are. This is sometimes called the vortex field.

(a) Suppose that we had a curve γ parameterised by

$$\vec{r}(\theta) = (f(\theta)\cos(\theta), f(\theta)\sin(\theta))$$

for $\theta \in [\theta_1, \theta_2]$ where f is a positive function. Show that $\vec{F} \cdot d\vec{r} = d\theta$ and conclude that

$$\int_\gamma \vec{F} \cdot d\vec{r} = \theta_2 - \theta_1.$$

Remember, given a parameterisation $\vec{r}(\theta)$, then $\vec{F} \cdot d\vec{r}$ means $\vec{F} \cdot \dfrac{d\vec{r}}{d\theta} d\theta$.

(b) Why is it true for a closed curve γ that wraps around origin n times (wrapping around in a positive sense is in the counterclockwise direction, and a negative sense in the clockwise direction) that we have

$$\int_\gamma \vec{F} \cdot d\vec{r} = 2\pi n.$$