Problem 1 – Parametrization Practice. We can describe a surface by a vector function

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

where u, v are two parameters. For example, the plane $x + y + z = 1$ could be described as the vector function

$$\mathbf{r}(x, y) = \langle x, y, 1 - x - y \rangle.$$

That same plane could also be described as

$$\mathbf{r}(y, z) = \langle 1 - y - z, y, z \rangle.$$

Find a parametric representation for each of the following surfaces:

(a) The plane $z = 1+2x+3y$ that lies above the rectangle $0 \leq x \leq 3, 0 \leq y \leq 2$

(b) The elliptic paraboloid $z = x^2 + 4y^2$

(c) The cylinder $x^2 + y^2 = 9$, $0 \leq z \leq 1$ (try cylindrical coordinates)

(d) The sphere $x^2 + y^2 + z^2 = a^2$ where a is a constant (try spherical coordinates)

(e) The "cone of shame" $z = \sqrt{x^2 + y^2}, 1 \leq z \leq 2$

Problem 2 – Tangent Planes to Surfaces. The equation of a tangent plane can be written as

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where (x_0, y_0, z_0) is a point on the plane and $\langle a, b, c \rangle$ is a normal vector to the plane. If we have a parametric representation $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ for a given surface, the normal to that surface is given by $\mathbf{r}_u \times \mathbf{r}_v$ where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$
$$\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.$$

Find the equation of a tangent plane to the parametrized surface

$$\mathbf{r}(u, v) = uv\mathbf{i} + u \sin v\mathbf{j} + v \cos u\mathbf{k}$$

at $u = 0, v = \pi$.

Problem 3 – Surface Area. If S is a smooth parametric surface described by a parametrization $\mathbf{r}(u, v)$, $(u, v) \in D$, then the surface area of S is given by

$$A = \iint_D ||\mathbf{r}_u \times \mathbf{r}_v|| dudv$$

Find the surface area of

(a) The sphere $x^2 + y^2 + z^2 = a^2$ where a is a constant (use your parametrization found in 1d)

(b) The "cone of shame" $z = \sqrt{x^2 + y^2}, 1 \leq z \leq 2$ (use your parametrization found in 1e)

Problem 4 – Scalar Surface Integrals. If S is a smooth parametric surface described by a parametrization $\mathbf{r}(u,v)$, $(u,v) \in D$, then the surface integral of a scalar function f over S is given by

$$\iint_S f dS = \iint_D f(\mathbf{r}(u,v))||\mathbf{r}_u \times \mathbf{r}_v||dudv$$

Evaluate the following surface integrals:

(a) $\iint_S x^2yzdS$ where S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $0 \leq x \leq 3, 0 \leq y \leq 2$ (use your parametrization found in 1a)

(b) $\iint_S ydS$ where S is the cylinder $x^2 + y^2 = 9$, $0 \leq z \leq 1$ (use your parametrization found in 1c)