Today:

- Order of convergence
- Comparing iterative methods

Homework Announcements:

- Make sure to “submit” your homework on CCLE, not just upload a draft
- Cite any important theorems you use and why you can use them
- For programming questions, make sure to write a short summary of your result (e.g. “My code estimated $p = 1.365230$ with a tolerance $10^{-6}$ and initial guess $p_0 = 1.5$”)
- Export images as .png or .jpg, don’t use the Matlab default .fig
- If you submitted a code but it has errors, make sure to fix before the exam
If \( p_n \) is a sequence converging to \( p \) with \( p_n \neq p \) for all \( n \), and if there exist \( \lambda, \alpha > 0 \) such that

\[
\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda,
\]

then \( p_n \) converges to \( p \) with order \( \alpha \).

\( \alpha = 1 \quad \Rightarrow \quad \) linear convergence

\( \alpha = 2 \quad \Rightarrow \quad \) quadratic convergence
Order of Convergence for Fixed-Point Iteration:

- Satisfy the hypothesis of the fixed-point theorem
- If \( p = g(p) \) and \( g'(p) \neq 0 \), then the fixed-point iteration converges **linearly**
- If \( p = g(p) \), \( g'(p) = 0 \), and \( g'' \) is bounded in an open interval around \( p \), then there is an interval where the fixed-point iteration converges **at least quadratically**
Example 1. Last week we showed the fixed-point iteration $x_{n+1} = g(x_n)$ with $g(x) = \arctan x + \frac{1}{2}$ converges on the interval $[1, 2]$ with some $x_0 \in [1, 2]$. What is the order of convergence?

\[ g(x) = \arctan x + \frac{1}{2} \quad g(x^*) = x^* \rightarrow \arctan x^* + \frac{1}{2} = x^* \]

\[ g'(x) = \frac{1}{1 + x^2} \quad g'(x^*) \neq 0 \rightarrow \text{expect linear convergence} \]

Claim $x_n$ converge linearly

Options for proving:
- cite theorem in textbook
- directly (Mean Value Theorem)
Example 1. Last week we showed the fixed-point iteration $x_{n+1} = g(x_n)$ with $g(x) = \arctan x + \frac{1}{2}$ converges on the interval $[1, 2]$ with some $x_0 \in [1, 2]$. What is the order of convergence?

$g \in C[1,2]$, $g$ is diff. on $(1,2)$

$|x_{n+1} - x^*| = |g(x_n) - g(x^*)|$

$= |g'(s_n)||x_n - x^*| \quad \text{by MVT where } s_n \text{ is between } x_n \text{ and } x^* \leq s_n < x_n$. We have $g'(s_n) = g'(x^*) = \frac{1}{1 + x^2} > 0$

Therefore $x_n \to x^*$ linearly.
Example 2. Show that the function $g(x) = \frac{2x^3 + 1}{3x^2}$ has the fixed point $x^* = 1$. Given that the fixed-point iteration $x_{n+1} = g(x_n)$ on the interval $[0.9, 1.1]$ converges to $x^* = 1$ with some $x_0 \in [0.9, 1.1]$, what is the order of convergence?

Fixed point: $g(x^*) = x^*$ i.e. $g(1) = 1$

$g(1) = \frac{2(1)^3 + 1}{3(1)^2} = \frac{2}{3} = 1 \checkmark$  $x^* = 1$ is a fixed point of $g$.

Scratch work:

$g(x) = \frac{2}{3}x + \frac{1}{3x^2}$

$g'(x) = \frac{2}{3} - \frac{2}{3}x^{-3}$  $g'(1) = \frac{2}{3} - \frac{2}{3} = 0$

$g''(x) = 2x^{-4}$  $g''(1) = 2$  quadratic convergence
Example 2. Show that the function \( g(x) = \frac{2x^3 + 1}{3x^2} \) has the fixed point \( x^* = 1 \). Given that the fixed-point iteration \( x_{n+1} = g(x_n) \) on the interval \([0.9, 1.1]\) converges to \( x^* = 1 \) with some \( x_0 \in [0.9, 1.1] \), what is the order of convergence?

Claim \( x_n \) converge quadratically.

Proof. \( g(x_n) < g(x^*) + g'(x^*) (x_n - x^*) + g''(x^*) (x_n - x^*)^2 \)

by Taylor's Theorem where \( \xi_n \) is between \( x_n \) and \( x^* \) (\( g \in C^2 [0.9, 1.1] \))

\[
|x_{n+1} - x^*| = |g(x_n) - g(x^*)| = |g''(\xi_n)| (x_n - x^*)^2
\]

\[
\lim_{n \to \infty} \frac{|x_{n+1} - x^*|}{(x_n - x^*)^2} = \lim_{n \to \infty} \frac{g''(\xi_n)}{2} = \frac{|g''(x^*)|}{2} = \frac{2}{2} = 1 > 0
\]

because \( \xi_n \) also converge to \( x^* \).

Therefore \( x_n \to x^* \) quadratically.
Note: it turns out the last example was Newton’s method with \( f(x) = x^3 - 1 \).

\[
x_{n+1} = g(x_n) = \frac{2x_n^3 + 1}{3x_n^2}
\]

Newton’s method:

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

\[
= x_n - \frac{x_n^3 - 1}{3x_n^2}
\]

\[
= \frac{3x_n^3 - x_n^3 + 1}{3x_n^2}
\]

\[
= \frac{2x_n^3 + 1}{3x_n^2}
\]
Order of Convergence for Newton’s Method:

- $f \in C^2[a, b], p \in [a, b]$
- If $f(p) = 0$ and $f'(p) \neq 0$, then Newton’s method converges \textbf{quadratically} provided we start close enough to $p$
- If $f(p) = 0, f'(p) = 0$, (i.e. $p$ is not a simple zero), then Newton’s method converges \textbf{linearly} provided we start close enough to $p$
**Example 3.** Consider Newton’s method of finding the root of $f(x) = 0$ where $f(x) = x^2(x - 1)$ with some initial guess $x_0$. What are the possible roots this method could converge to? What is the order of convergence we would expect for each of these roots?

$$f(x) = x^3 - x^2$$
$$f'(x) = 3x^2 - 2x$$

For $x = 0$:
- $x^* = 0$
- $x^* = 1$
- $\text{mult.} 2$
- $\text{mult.} 1$
- $\text{not simple}$
- $\text{simple zero}$

- **Expect linear convergence near** $x = 0$
- **Expect quadratic convergence near** $x = 1$

For $x = 1$:
- $f'(0) = 3(0)^2 - 2(0) = 0$
- **Not simple**
- $f'(1) = 3(1)^2 - 2(1) = 1$
- **Simple**
For the last example, we run `example3.m` and `newton.m` with a tolerance of $10^{-16}$.

- If we start with $x_0 = 0.1$, our method converges to $x = 0$ but it takes 24 iterations!
- If we start with $x_0 = 1.4$, our method converges to $x = 1$ in only 6 iterations.
Example 4. Consider Newton’s method of finding the root of \( f(x) = 0 \) where \( f(x) = x^2 + 2xe^{-x} + e^{-2x} \) with some initial guess \( x_0 \in [0, 1] \). Does this method converge linearly or quadratically?

\[
f(x) = (x + e^{-x})^2
\]

\[
f'(x) = 2(x + e^{-x})(1 - e^{-x})
\]

\[
f''(x^*) = 2(x^* + e^{-x^*})(1 - e^{-x^*}) = 0
\]

(\text{linear convergence})
Example 5. Compare the Bisection, Fixed-Point, Newton’s, Secant, and False Position methods for finding the root of $x^3 + 4x^2 - 10 = 0$ on $[1, 2]$ with a tolerance $10^{-6}$.

- See example5.m, bisection.m, fixedpoint.m, newton.m, secant.m, falseposition.m
- Bisection, Newton, Secant, and False Position methods all use $f(x) = x^3 + 4x^2 - 10$
- Newton’s method also requires $f'(x) = 3x^2 + 8x$
- Fixed-Point method uses $g(x) = \sqrt{\frac{10}{4+x}}$
- Initial guess for Fixed-Point and Newton methods: $p_0 = 1.5$
- Initial guesses for Secant and False Position methods: $p_0 = 1.25, p_1 = 1.5$ ($f(p_0) < 0, f(p_1) > 0$)

Fixed point: $x^3 + 4x^2 - 10 = 0$

$x^3 + 4x^2 + x - 10 = x \Rightarrow g_1(x) = x^3 + 4x^2 + x - 10$ diverges $g_1'(x) = 3x^2 + 8x + 1$ on $(1, 2)$
### Summary:

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<thead>
<tr>
<th></th>
<th>Bisection</th>
<th>Fixed-Point</th>
<th>Newton’s</th>
<th>Secant</th>
<th>False Position</th>
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<tbody>
<tr>
<td><strong>Solves for</strong></td>
<td>( f(x^*) = 0 )</td>
<td>( g(x^<em>) = x^</em>   )</td>
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<td><strong>Order of conv.</strong></td>
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<td><strong>Pros</strong></td>
<td>always converges</td>
<td>error analysis</td>
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<td><strong>Cons</strong></td>
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<td>might not converge</td>
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<td>require a contraction</td>
<td>need ( f' ) close to 0</td>
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